

# MOEBIUS TRANSFORMATIONS PRESERVING FIXED ANHARMONIC RATIO

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ABSTRACT. O. Kobayashi [6] proved that  $C^1$ -mappings preserving anharmonic ratio are Moebius transformations. We strengthen his result and prove, that the requirement of differentiability and even of injectivity can be omitted.

## 1. INTRODUCTION

A concept of Apollonian tetrad in the complex plane  $\mathbf{C}$  was introduced in the paper ([1], Def. 1, p.15): any ordered quadruple of distinct points  $\{z_1, z_2, z_3, z_4\} \subset \mathbf{C}$  is called an *Apollonian* tetrad, if

$$|z_2 - z_3| \cdot |z_1 - z_4| = |z_3 - z_1| \cdot |z_2 - z_4| = |z_1 - z_2| \cdot |z_3 - z_4| \quad (0.1)$$

It was proved in ([1], Main Theorem, p. 19), that any univalent analytic function in the domain  $D \subset \mathbf{C}$  is linear-fractional iff the image of any Apollonian tetrad in  $D$  is also Apollonian tetrad.

Since that time more than ten articles appeared, in which different generalizations of the mentioned property of tetrad to finite set of points were introduced: Haruki H. and Rassias T.M. in [2] considered Apollonian triangles and hexagons, Bulut S. and Yilmaz Özgür N. in [3] considered Apollonian set consisting of  $2n$  pairwise different points and proved, that the analytic univalent function is linear-fractional iff the image of any Apollonian set in  $D$  is also an Apollonian set.

The constructions in these works have considerable interest from the point of view of the theory of quasimöbius transformations in  $\overline{\mathbf{R}}^n$  (see [4]) and in Ptolemaic spaces (see [5]).

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O. Kobayashi [6] noticed, that relation (0.1) is equivalent to the equality  $[z_1 : z_2 : z_3 : z_4] = (1 \pm i\sqrt{3})/2$ , where  $[z_1 : z_2 : z_3 : z_4]$  is the anharmonic ratio of the quadruple  $\{z_1, z_2, z_3, z_4\}$ , and he obtained the following result:

**Theorem 1.** ([6], Theorem 2.1, p. 118)

*Let  $\lambda \in \mathbf{C} \setminus \mathbf{R}$  and  $U \subset \mathbf{C}$  be a domain. Suppose  $f : U \rightarrow \mathbf{C}$  is an injective differentiable mapping.*

*If for any quadruple of pairwise distinct points  $\{z_1, z_2, z_3, z_4\} \in U$  with anharmonic ratio  $[z_1 : z_2 : z_3 : z_4] = \lambda$  the equality  $[f(z_1) : f(z_2) : f(z_3) : f(z_4)] = \lambda$  holds, then  $f$  is a Moebius transformation.*

In this paper we show, that the theorem is valid for any  $\lambda \in \overline{\mathbf{C}} \setminus \{0, 1, \infty\}$  and for any continuous non-constant mapping  $f : U \rightarrow \overline{\mathbf{C}}$  of the domain  $U \subset \overline{\mathbf{C}}$  without the requirement of injectivity and differentiability of  $f$ .

## 2. THE TABLES OF SYMBOLS, TERMINOLOGY AND THE MAIN RESULT

Let  $\mathbf{T}$  be set of all ordered quadruples (tetrads)  $T = \{z_1, z_2, z_3, z_4\}$  in the extended complex plane  $\overline{\mathbf{C}}$ , that do not contain any three coincident elements.

A tetrad with four pairwise different elements is called *nonsingular*.

For any tetrad we define its anharmonic ratio (see [7], §44)  $A(T) = [z_1 : z_2 : z_3 : z_4]$ .

If all points in a tetrad  $T$  are finite and pairwise different, then  $A(T)$  is

$$A(T) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_3 - z_2)(z_4 - z_1)}. \quad (1.1.1)$$

Under these conditions  $A(T)$  is different from 0, 1 and  $\infty$ . For tetrads  $s_{12}(T)$ ,  $s_{13}(T)$ ,  $s_{14}(T)$ , obtained from  $T$  by permutation of first and second, of first and third, of first and fourth elements respectively the following equalities hold:

$$A(s_{12}(T)) = \frac{1}{A(T)}; \quad A(s_{13}(T)) = \frac{A(T)}{A(T) - 1}; \quad A(s_{14}(T)) = 1 - A(T) \quad (1.1.2)$$

If in the tetrad  $T = \{z_1, z_2, z_3, z_4\}$  neither three elements coincide, there exists a finite or infinite limit

$$A(T) = \lim_{w_1 \rightarrow z_1, w_2 \rightarrow z_2, w_3 \rightarrow z_3, w_4 \rightarrow z_4} \frac{(w_1 - w_3)(w_2 - w_4)}{(w_3 - w_2)(w_4 - w_1)}, \quad (1.1.3)$$

where the limit is taken on a set of all nonsingular tetrads  $\{w_1, w_2, w_3, w_4\}$ . This limit defines the anharmonic ratio of the tetrad  $T$  in general case. Particularly, for nonsingular tetrad  $T = \{z_1, z_2, z_3, \infty\}$  we have  $A(T) = -(z_1 - z_3)/(z_3 - z_2)$ . We notice, that the tetrad  $T \in \mathbf{T}$  is nonsingular iff  $A(T)$  different from 0, 1 and  $\infty$ .

A Moebius transformation  $\mu : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$  is defined as a superposition of finite number of reflections (or inversions) with respect to generalized circles in  $\overline{\mathbf{C}}$  (see [7], Def 3.1.1, P. 25) and it realized either by the linear-fractional function or by its conjugate.

If  $\mu$  is a linear-fractional mapping, then for any tetrad  $T \in \mathbf{T}$  the equality  $A(\mu(T)) = A(T)$  (the invariance of anharmonic ratio by the linear-fractional mappings) holds. The opposite is also true: a bijective mapping  $\mu : \mathbf{C} \rightarrow \mathbf{C}$ , which preserves an anharmonic ratio of all nonsingular tetrads, is realized by linear-fractional function (see [7], §4.4).

For given complex number  $\alpha \notin \{0, 1, \infty\}$ , we denote by  $\mathbf{T}(\alpha)$  the set of all tetrads  $T \in \mathbf{T}$  with  $A(T) = \alpha$  (all that tetrads are nonsingular) and by  $\mathbf{T}(\alpha, f)$  the set of all tetrads of  $\mathbf{T}(\alpha)$ , for which  $f(T) \in \mathbf{T}$ .

**Definition 2.** We say that a mapping  $f : U \rightarrow \overline{\mathbf{C}}$ , of a domain  $U \subset \overline{\mathbf{C}}$  satisfies the condition  $\varphi(K, \alpha)$ , if for any tetrad  $T \in \mathbf{T}(\alpha, f)$  the equality  $A(f(T)) = \alpha = A(T)$  holds.

The main result of the paper is the following

**Theorem 3.** If  $\alpha \notin \{0, 1, \infty\}$ , then any continuous mapping  $f : U \rightarrow \overline{\mathbf{C}}$  of a domain  $U \subset \overline{\mathbf{C}}$ , which satisfies the condition  $\varphi(K, \alpha)$ , is either a constant or the Moebius transformation; and if  $\alpha$  is not a real number, then a function  $f(z)$  is either a constant or a linear-fractional function.

## 3. THE INJECTIVITY LEMMA

**Proposition 4.** *If a continuous mapping  $f : U \rightarrow \overline{\mathbf{C}}$  satisfies the condition  $\varphi(K, \alpha)$ ,  $\alpha \notin \{0, 1, \infty\}$ , then it satisfies the condition  $\varphi(K, \beta)$ , where  $\beta$  is taken from the set*

$$P_\alpha = \left\{ \alpha, \frac{1}{\alpha}, 1 - \alpha, \frac{1}{1 - \alpha}, \frac{\alpha}{\alpha - 1}, \frac{\alpha - 1}{\alpha} \right\} \quad (2.1.1)$$

**Proof.** The proof follows immediately from relations (1.1.2) and from the fact, that the equality  $f(s(T)) = s(f(T))$  is preserved by the permutations  $s$  of the elements of the tetrad, and therefore the relations  $f(T) \in \mathbf{T}$  and  $f(s(T)) \in \mathbf{T}$  are equivalent. ■

**Lemma 5.** *Suppose  $\alpha \notin \{0, 1, \infty\}$  and let  $f : U \rightarrow \overline{\mathbf{C}}$  be a continuous mapping of a domain  $U \subset \overline{\mathbf{C}}$  satisfying the condition  $\varphi(K, \alpha)$ . Then  $f(z)$  is either injective in  $U$  or  $f$  is a constant map.*

**Proof.** We assume that  $f$  is not injective. Then we can find different points  $\zeta_0, \zeta_\infty \in U$ , for which  $f(\zeta_0) = f(\zeta_\infty) = a \in \overline{\mathbf{C}}$ . Take such linear-fractional mappings  $\mu$  and  $\nu$  that  $\mu(\zeta_\infty) = \nu(a) = \infty$ . Evidently  $\infty \in \mu(U) = U'$ . A continuous mapping  $g = \nu \circ f \circ \mu^{-1} : U' \rightarrow \overline{\mathbf{C}}$  satisfies the condition  $\varphi(K, \gamma)$  for any constant  $\gamma \in P_\alpha$ ; whereas  $g(\infty) = g(\mu(\zeta_\infty)) = \infty$ .

Consider the set  $M = g^{-1}(\{\infty\}) \cap \mathbf{C} = \{z \in \mathbf{C} : g(z) = \infty\}$ .

Since  $g$  is continuous, the set  $M$  is closed in  $U' \setminus \{\infty\}$ . We show, that  $M$  is an open set.

Let  $z_0$  be an any point in  $M$ .

Build an open disc  $B = B(z_0, r) \subset U'$  with radius

$$r = \frac{\max\{1, \text{dist}(z_0, \partial U')\}}{2 + |\alpha|} \quad (2.2.1)$$

If  $g(z) \equiv \infty$  in  $B$ , then  $B \subset M$ , and  $z_0$  is an interior point of the set  $M$ . Consider the point  $z_1 \in B$ , in which  $g(z_1) \neq \infty$ . The point  $z_2 = z_0 + \alpha(z_1 - z_0)$  lies in  $U'$ , whereas

$$|z_2 - z_0| = |\alpha| \cdot |z_1 - z_0| \leq \text{dist}(z_0, \partial U') \frac{|\alpha|}{2 + |\alpha|} < \text{dist}(z_0, \partial U').$$

For  $T = \{z_2, z_1, z_0, \infty\} \subset U'$  we have  $A(T) = -(z_2 - z_0)/(z_0 - z_1) = \alpha$ . The assumption, that  $f(T) = \{g(z_2), g(z_1), \infty, \infty\} \in \mathbf{T}$ , contradicts

the condition  $\varphi(K, \alpha)$ , because in this case  $A(f(T)) \in \{0, 1, \infty\}$ , that is  $f(A) \neq \alpha$ .

Therefore the tetrad  $f(T)$  has three equal elements. But  $g(z_1) \neq \infty$  implies that  $g(z_2) = \infty$ , that is  $z_2 \in M$ .

Since  $g(z_1) \neq \infty$  and  $f$  is continuous in the point  $z_1$ , take such  $\varepsilon > 0$  that the disc  $B(z_1, \varepsilon) \subset B$  and a function  $g(z) \neq \infty$  in  $B(z_1, \varepsilon)$ . From the inequality  $\varepsilon < r$  and from (2.2.1) it follows that

$$B\left(z_0, \frac{|\alpha|}{1+|\alpha|}\right) \subset B \quad (2.2.2)$$

Putting  $\delta = \varepsilon \frac{|\alpha|}{1+|\alpha|}$ , consider any point  $z \in B(z_0, \delta)$ . For  $w = z + \alpha^{-1}(z_2 - z)$  the inequality holds:

$$|w - z_1| = |z - z_1 + z_1 - z_0 + \alpha^{-1}(z_0 - z)| < |z - z_0| \left|1 - \frac{1}{\alpha}\right| < \varepsilon,$$

which means, that  $w \in B(z_1, \varepsilon)$  and therefore  $g(w) \neq \infty$ . For  $T' = \{z_2, w, z, \infty\} \subset U'$  we have  $A(T') = -(z_2 - z)/(z - w) = \alpha$ . The assumption that  $g(T') = \{g(z_2) = \infty, g(w), g(z), \infty\} \in \mathbf{T}$  contradicts the condition  $\varphi(K, \alpha)$ , because  $A(g(T')) \in \{0, 1, \infty\}$  and therefore  $A(g(T')) \neq \alpha$ . Then in the tetrad  $g(T')$  there are three equal elements. But  $g(w) \neq \infty$  and therefore  $g(z) = \infty$ . Thus we see, that  $g(z) \equiv \infty$  in the disc  $B(z_0, \delta)$ . So  $B(z_0, \delta) \subset M$  and  $z_0$  is an interior point of the set  $M$ . Since any point of the set  $M$  it is an interior point, the set  $M$  is open.

Since the set  $U' \setminus \{\infty\}$  is connected, the open-and-closed set  $M$  is an empty set or  $U' \setminus \{\infty\}$ . But  $\mu(\zeta_0) \in M$ , the  $M \neq \emptyset$ . Therefore  $M = U' \setminus \{\infty\}$ , and it means that  $g(z) \equiv \infty$  in  $U'$ . Thus  $f(\zeta) \equiv a$  in  $U$ . ■

#### 4. PROOF OF THE MAIN THEOREM

We prove the main theorem in several steps arranging them as independent propositions.

**Proposition 6.** *Suppose the domain  $U \subset \overline{\mathcal{C}}$  contains  $\infty$ , and a continuous injective mapping  $f : U \rightarrow \overline{\mathcal{C}}$ ,  $f(\infty) = \infty$  satisfies the condition*

$\varphi(K, \alpha)$ ,  $\alpha \notin \{0, 1, \infty\}$ . Then for any  $z_0 \in U$  and  $a_0 = z_0 + w_0$ ,  $b_0 = z_0 - w_0$ , where  $|w_0| < \text{dist}(z_0, \partial U)$ , the equality holds:

$$f(z_0) = \frac{f(a_0) + f(b_0)}{2} \quad (3.1.1)$$

**Proof.** Let  $\alpha \in \{2, 1/2, -1\}$ . Then by Proposition 4 the mapping  $f$  satisfies the condition  $\varphi(K, 2)$ . For  $T = \{a_0, b_0, z_0, \infty\}$  we obtain

$$2 = A(T) = [f(a_0) : f(z_0) : f(b_0) : \infty] = -\frac{f(a_0) - f(b_0)}{f(b_0) - f(z_0)},$$

from which the desired equation (3.1.1) immediately follows.

Suppose now  $\alpha \notin \{0, 1, \infty, 2, 1/2, -1\}$ .

If  $|\alpha| > 1$ , then we take  $\beta = \alpha$ ; in otherwise we take  $\beta = 1/\alpha$ . Then  $f$  satisfies the condition  $\varphi(K, \beta)$ ,  $\beta \notin \{0, 1, \infty, 1/2\}$  (see proposition 4). For  $\beta' = (1 - \beta)/(1 - 2\beta)$  we have the equation  $(1 - 2\beta)/(1 - 2\beta') = -1$ . As  $\beta \notin \overline{B}(1/2, 1/2)$ , then  $|\beta - 1/2| > 1/2$ , that is  $|1 - 2\beta| > 1$ . Therefore  $|1 - 2\beta'| = q < 1$ .

Put  $R = \text{dist}(z_0, \partial U)$  and  $w_k = (2\beta' - 1)^k w_0$ ,  $k = 1, 2, \dots$

As  $|w_k| = q^k |w_0| < R$ , then all points  $a_k = z_0 + w_k$  and  $b_k = z_0 - w_k$  lies in the disc  $B(z_0, R) \subset U$ . We show that for any  $k = 0, 1, 2, \dots$  the follow equality is valid:

$$f(a_k) + f(b_k) = f(a_0) + f(b_0) \quad (3.1.2)$$

For  $k = 0$  (3.1.2) is trivial. We suggest, that it holds for some  $k$  and show, that  $f(a_{k+1}) + f(b_{k+1}) = f(a_0) + f(b_0)$ .

For the tetrad  $T_1 = \{b_k, a_{k+1}, b_{k+1}, \infty\}$  we have

$$A(T_1) = -\frac{b_k - b_{k+1}}{b_{k+1} - a_{k+1}} = -\frac{w_{k+1} - w_k}{-2w_{k+1}} = \frac{\beta' - 1}{2\beta' - 1} = \beta.$$

The equality

$$A(T_1) = -\frac{f(b_k) - f(b_{k+1})}{f(b_{k+1}) - f(a_{k+1})} = \beta$$

follows from the condition  $\varphi(K, \beta)$ . For  $T_2 = \{a_k, b_{k+1}, a_{k+1}, \infty\}$  we have

$$A(T_2) = -\frac{a_k - a_{k+1}}{a_{k+1} - b_{k+1}} = -\frac{w_k - w_{k+1}}{2w_{k+1}} = \beta.$$

Therefore

$$A(T_2) = -\frac{f(a_k) - f(a_{k+1})}{f(a_{k+1}) - f(b_{k+1})} = \beta.$$

Thus we come to the equality

$$\frac{f(b_{k+1}) - f(b_k)}{f(b_{k+1}) - f(a_{k+1})} = \frac{f(a_k) - f(a_{k+1})}{f(b_{k+1}) - f(a_{k+1})},$$

from which it follows that  $f(b_{k+1}) - f(b_k) = f(a_k) - f(a_{k+1})$ .

Then by induction we obtain

$$f(a_{k+1}) + f(b_{k+1}) = f(a_k) + f(b_k) = f(a_0) + f(b_0).$$

Thus we have proved (3.1.2) for all  $k = 0, 1, 2, \dots$

Since  $w_k \rightarrow 0$ , when  $k \rightarrow \infty$ , then  $a_k \rightarrow z_0$  and  $b_k \rightarrow z_0$ .

Next from (3.1.2) we obtain the desired relation (3.1.1). ■

**Proposition 7.** *Suppose a domain  $U \subset \overline{\mathcal{C}}$  contains  $\infty$ , and a continuous mapping  $f : U \rightarrow \overline{\mathcal{C}}$ ,  $f(\infty) = \infty$  satisfies the condition  $\varphi(K, \alpha)$ ,  $\alpha \notin \{0, 1, \infty\}$ . Then the mapping  $f$  moves any linear segment  $L \subset U \setminus \{\infty\}$  to a linear segment; any ray  $P \subset U \setminus \{\infty\}$  to some ray; any line  $Q \subset U \setminus \{\infty\}$  to some line.*

**Proof.** We show that for any point  $c \in U \setminus \{\infty\}$  the mapping  $f$  is linear on any linear segment  $L \subset B(c, r)$ , where  $r = \text{dist}(c, \partial U)/3$ . Let  $a, b$  be the endpoints of the segment  $L$ . For any point  $z \in L$ ,  $\max\{|z - a|, |z - b|\} \leq 2r < \text{dist}(z, \partial U)$ . That is, we may apply the Proposition 6 to any point  $z \in L$  and any  $w$  such that  $|w| < \max\{|z - a|, |z - b|\}$ . Therefore for any pair of points  $t_1, t_2 \in L$  we have  $f((t_1 + t_2)/2) = (f(t_1) + f(t_2))/2$ . We conclude that the function  $f$  is linear on a dense subset of the segment  $L$  and so, by continuity of  $f$ , it is linear on  $L$ .

Thus the mapping  $f$  is locally linear on any connected subset  $S \subset U \setminus \{\infty\}$ , which lies on a straight line. Therefore by connectedness of  $S$ ,  $f$  is linear on all of  $S$ . Particularly, an image of any segment is some segment, the image of any ray is some ray, the image of any line is some line. ■

Next we use a criterion of Moebiusness for mappings of  $n$ -dimensional domains, proposed by Zelinsky Y.B. for the mapping of (we take this theorem for case  $n = 2$ ).

**Theorem 8.** ([8], Th. 8, P. 35) *Suppose a continuous mapping  $f : D \rightarrow \overline{\mathcal{C}}$  of a domain  $D \subset \overline{\mathcal{C}}$  moves any set  $P \subset D$ , which lies on*

a generalized circle to a set, which lies on a generalized circle. If  $f(D)$  does not lie on a generalized circle, then  $f$  is a Moebius transformation.

**Lemma 9.** *Any continuous injective mapping  $f : U \rightarrow \overline{\mathbf{C}}$  of a domain  $U \subset \overline{\mathbf{C}}$ , which satisfies the condition  $\varphi(K, \alpha)$ ,  $\alpha \notin \{0, 1, \infty\}$ , is a Moebius transformation.*

**Proof.** Take any open disc  $D \subset U$  such that  $\overline{D} \subset U$  and a set  $P \subset D$  lies on a generalized circle  $S \subset \overline{\mathbf{C}}$ . Then  $S \cap D$  is a connected subset of a generalized circle  $S$ .

The situation 1.

Suppose  $S \cap \partial D \neq \emptyset$  and  $a \in S \cap \partial D$ . We build linear-fractional mappings  $\mu : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$  and  $\eta : \overline{\mathbf{C}} \rightarrow \overline{\mathbf{C}}$  such that  $\mu(a) = \infty$ ,  $\eta(f(a)) = \infty$  and consider the mapping  $g = \eta \circ f \circ \mu^{-1} : \mu(U) \rightarrow \overline{\mathbf{C}}$ . This mapping satisfies the condition  $\varphi(K, \alpha)$  and  $\infty \in \mu(U)$ ,  $g(\infty) = \infty$ . The set  $L' = \mu((S \cap \overline{D}) \setminus \{a\}) \subset \mu(U) \setminus \{\infty\}$  is either a ray or a line. By Proposition 7 the set  $g(L')$  is also either some ray or a line. Therefore  $f = \eta^{-1} \circ g \circ \mu$  maps a set  $S \cap D$  (and any subset  $P$  also) to a subset of a circle.

The situation 2.

Take  $S \subset D$  and  $a \in S$ . We build linear-fractional mappings  $\mu$  and  $\eta$  by analogy with the situation 1 and consider the mapping  $g = \eta \circ f \circ \mu^{-1}$ . By Proposition 7 the mapping  $g$  moves the line  $L' = \mu((S \setminus \{a\}) \subset \mu(U) \setminus \{\infty\}$  to some line. Therefore  $g(L')$  is a generalized circle and  $f$  moves the generalized circle  $S = \mu^{-1}(L')$  to a generalized circle, and any subset  $P \subset S$  to a subset of a generalized circle. Thus  $f$  satisfies the conditions of the theorem 8 in the disc  $D$  and is injective. Therefore  $f$  is a Moebius transformation on the disc  $D$ . Since our choice of the disc  $D \subset U$  is arbitrary, the mapping  $f$  is locally Moebius on a domain  $U$ . Moebius transformations are explicitly defined by their values on some quadruple, that does not lie on a generalized circle. From local Moebiusness of  $f$  it follows, that  $f$  is Moebius on a domain  $U$ . ■

**Lemma 10.** *Any continuous injective mapping  $f : U \rightarrow \overline{\mathbf{C}}$  of a domain  $U \subset \overline{\mathbf{C}}$ , which satisfies the condition  $\varphi(K, \alpha)$ ,  $\alpha \in \mathbf{C} \setminus \mathbf{R}$  is a linear-fractional function.*



**Proof.** By Lemma 9 the mapping  $f$  is a Moebius transformation, so  $f(z)$  is either a linear-fractional function or its conjugate. We show that if  $\alpha = a + ib$  and  $b \neq 0$ , then the second is impossible. Let  $f(z) = \overline{\mu(z)}$ , where  $\mu(z)$  is a linear-fractional function. Take any tetrad  $T = \{z_1, z_2, z_3, z_4\} \subset U$  with anharmonic ratio  $A(T) = \alpha = a + ib$ . Then  $A(\mu(T)) = A(T) = a + ib$  and  $A(f(T)) = A(\overline{\mu(T)}) = \overline{A(\mu(T))} = a - ib \neq \alpha$ . This contradicts the condition  $\varphi(K, \alpha)$ . Therefore  $f(z)$  is a linear-fractional function. ■

The proof of the main Theorem follows immediately from Lemma 5, Lemma 9 and Lemma 10.

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